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By
Dr. Amit Prakash
P.G. Deptt. of Mathematics

Jordan Holder Theorem

The Jordan Holder theorem is a fundamental results in group theory, which deals the structure of finite groups. It essentially states that any two composition series of a finite group are equivalent in a certain sense.

Statement

Let G be a finite group then any to composition series are equivalent.

Proof :- We will prove the theorem by useing induction on $O(G)$.

If $O(G) = n$ if $n = 2$ then G has only composition series.

$$(e) = G_0 \Delta G_1 = G$$

So the theorem is hold for 2

Suppose that the theorem is true for $n > 2$ and every group of order less than n

Let two composition series

$$G = G_0 \Delta G_1 \Delta G_2 \Delta \cdots \Delta G_n = \{e\} \quad -(1)$$

$$G = H_0 \Delta H_1 \Delta H_2 \Delta \cdots \Delta H_n = \{e\} \quad -(2)$$

We consider two cases

(I) $G_1 = H_1$ & (II) $G_1 \neq H_1$

Case -(I) If $G_1 = H_1$

$\because O(G_1) < O(G)$ By induction hypothesis the two composition series

$$G_1 = G_1 \Delta G_2 \Delta G_3 \Delta \cdots\cdots \Delta G_n = \{e\}$$

$$G_1 = H_1 \Delta H_2 \Delta H_3 \Delta \cdots\cdots \Delta H_n = \{e\}$$

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Cont. from (P-4)

If G_1 are equivalent, this shows that (1) and (2) are equivalent

Case - (ii) If $G_1 \neq H_1$

$G_1 H_1$ is normal subgroup of G containing both G_1 and H_1

Also G_1 and H_1 are maximal normal subgroup of G.

We must have

$$G_1 H_1 = G$$

because otherwise

$$G_1 H_1 = G_1 = H_1$$

By second isomorphic theorem

$$\frac{G}{G_1} = \frac{G_1 H_1}{G_1} \cong \frac{H_1}{G_1 \cap H_1} \cong \frac{H_1}{K_2} \quad (\text{where } G_1 \cap H_1 = K_2)$$

$$\frac{G}{H_1} = \frac{H_1 G_1}{H_1} \cong \frac{G_1}{G_1 \cap H_1} \cong \frac{G_1}{K_2}$$

$\because$ $\frac{G}{G_1}$ and $\frac{G}{H_1}$ are simple group

the group $\frac{H_1}{K_2}$ and $\frac{G_1}{K_2}$ are also simple

Consider a composition series of K_2 as

$$K_2 \triangle K_3 \triangle K_4 \triangle \cdots \triangle K_t = \{e\}$$

then the series

$$G \triangle G_1 \triangle K_2 \triangle K_3 \cdots \triangle K_t = \{e\} \quad -(3)$$

$$G \triangle H_1 \triangle K_2 \triangle K_3 \cdots \triangle K_t = \{e\} \quad -(4)$$

are both composition series of G

Now (1) and (3) are equivalent by case-(i)

Cont. from (P-5)

Similarly (2) and (4) are equivalent
Now (3) and (4) are themselves equivalent, the quotients in these cases are

$$\frac{G}{G_1}, \frac{G_1}{K_2}, \frac{K_2}{K_3} \cdots \frac{K_{t-1}}{K_t}$$

$$\frac{G}{H_1}, \frac{H_1}{K_2}, \frac{K_2}{K_3} \cdots \frac{K_{t-1}}{K_t}$$

the first two quotients group are isomorphic in reverse order.

$\because$ for composition series, the relation of being equivalent is an equivalance relation.

Thus we get (1) and (2) are equivalent.